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A BANACH SPACE WITH, UP TO EQUIVALENCE, PRECISELY TWO SYMMETRIC BASES

BY

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ABSTRACT

It is well known that the classical sequence spaces c_0 and l_p $(1 \leqq p < \infty)$ have, up to equivalence, just one symmetric basis. On the other hand, there are examples of Orlicz sequence spaces which have uncountably many mutually non-equivalent symmetric bases. Thus in [4], p. 130, the question is asked whether there is a Banach space with, up to equivalence, more than one symmetric basis, but not uncountably many. In this paper we answer the question positively, by exhibiting a Banach space with, up to equivalence, precisely two symmetric bases.

§1. Introduction

(1) If X is a Banach space with a normalised symmetric basis $(x_n)_{n=1}^{\infty}$ with symmetric constant 1, and $\sigma = \{\sigma_j\}_{j=1}^{\infty}$ is a sequence of consecutive disjoint finite subsets of the integers, denote by $\bar{\bar{\sigma}}_j$ the number of elements in σ_j, and define the "averaging projection" P_σ on X as follows:

$$P_\sigma(x) = \sum_{j=1}^{\infty} \left(\left(\sum_{n \in \sigma_j} a_n \right) \Big/ \bar{\bar{\sigma}}_j \right) \left(\sum_{n \in \sigma_j} x_n \right) \qquad \left(x = \sum_{n=1}^{\infty} a_n x_n \in X \right).$$

Then we quote from [4], p. 117:

If X is a Banach space with normalised symmetric basis $\{x_n\}_{n=1}^{\infty}$, and P_σ is an averaging projection $X \to X$, put

$$U = P_\sigma(X).$$

Then $X \cong X \oplus U$.

(2) Our method of producing a space with just two symmetric bases is to produce spaces Y and Z, each with a symmetric basis, so that there is an

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averaging projection P on Y with $PY \cong Z$, and an averaging projection Q on Z such that $QZ \cong Y$.

It follows from the proposition that

$$Y \cong Y \oplus Z \quad \text{and} \quad Z \cong Z \oplus Y.$$

Thus $Y \cong Z$. So Y has at least two non-equivalent symmetric bases. We then, in the remainder of the paper, prove that, up to equivalence, the only other symmetric basic sequence in Y is the unit vector basis of l_1. Since our space is obviously not equivalent to l_1, this completes the proof.

(3) First we introduce some notation. Let us choose, once and for all, a bijection

$$i \to (\phi(i), \psi(i)) : \mathbf{N} \to \mathbf{N} \times \mathbf{N}.$$

Then if X is any Banach space with chosen normalised symmetric basis $(e_i)_{i=1}^{\infty}$, and $x, y \in X$, we define

(a) $x : y = \sum_{i=1}^{\infty} x_{\phi(i)} y_{\psi(i)} e_i$ $(x = \sum_{i=1}^{\infty} x_i e_i,\ y = \sum_{i=1}^{\infty} y_i e_i)$.

(b) $[x, y] = \sum_{i=1}^{\infty} z_i e_i$, where $z_{2i} = y_i$, $z_{2i-1} = x_i$ for each $i \in \mathbf{N}$.

If π is a permutation on $\mathbf{N}$ (written $\pi \in S(\mathbf{N})$) we define

(c) $\pi(x) = \sum_{i=1}^{\infty} x_{\pi(i)} e_i$,

also

(d) $\hat{x} = \pi(|x|)$,

where π is chosen so that the coordinates of $\hat{x}$, with respect to $(e_i)_{i=1}^{\infty}$, are monotonic decreasing.

These definitions of course depend on the choice of symmetric basis; but it will always be clear what basis we are using.

Considering definition (a), we see that, in the general case of a Banach space with symmetric basis, there is no guarantee that just because x, y are in X, $x : y$ will be in X. However, we shall in fact be using spaces which not only do have this property, but also satisfy the very strong inequality

$$\| x : y \|_X \leqq \| x \|_{c_0} \| y \|_X + \| y \|_{c_0} \| x \|_X.$$

It is by this inequality that we prove that there are only three symmetric basic sequences in our space (up to equivalence).

NOTE. Z. Altshuler, in [1], uses techniques somewhat similar to those in this paper, to produce a space with symmetric basis which contains no c_0 or l_p, and all its symmetric basic sequences are equivalent. His norm satisfies

$$\| x : y \|_X \leqq C(\| x \|_{c_0} \| y \|_X + \| y \|_{c_0} \| x \|_X), \qquad \text{with } C > 1.$$

It is not clear that there are norms not equivalent to the c_0 norm, which satisfy the stronger inequality: our first task is to produce some.

§2. Producing some special norms

Let $X = l_1$. For the purposes of the definitions (a) to (d) in §1, the symmetric basis we use is the canonical unit vector basis $(e_i)_{i=1}^{\infty}$. We define a sequence of norms $\|\cdot\|_{(n)}$ inductively, as follows:

$$\|x\|_{(1)} = \|x\|_{l_1};$$

$$\|x\|_{(i+1)} = \inf\Big\{ \|x_1\|_{l_1} + \sum_{r=1}^{k} (\|y_r\|_i \|z_r\|_{c_0} + \|y_r\|_{c_0} \|z_r\|_i) \text{ such that there are } \pi_1, \pi_2, \cdots, \pi_k \in S(\mathbf{N}) \text{ with } |x| \leqq x_1 + \sum_{r=1}^{k} \pi_r(y_r : z_r) \Big\}.$$

We then put

$$\|x\| = \lim_{i\to\infty} \|x\|_i = \inf_i \|x\|_i.$$

This norm is a symmetric norm on l_1 satisfying

$$\|x\|_{c_0} \leqq \|x\| \leqq \|x\|_{l_1},$$

and

$$\|x\| = \inf\Big\{ \|x_1\|_{l_1} + \sum_{r=1}^{k} (\|y_r\|_{c_0} \|z_r\| + \|y_r\| \|z_r\|_{c_0}) \text{ such that } |x| \leqq x_1 + \sum_{r=1}^{k} \pi_r(y_r : z_r) \Big\}.$$

In particular,

$$\|x : y\| \leqq \|x\|_{c_0} \|y\| + \|y\|_{c_0} \|x\|.$$

In fact, $\|\cdot\|$ is the largest symmetric norm on l_1 less than $\|\cdot\|_{l_1}$ which has this property. We wish to prove that $\|\cdot\|$ is not equivalent to $\|\cdot\|_{c_0}$. We do this by showing that, as $k \to \infty$,

$$v_k = \Big\| \sum_{i=1}^{k} e_i \Big\| \to \infty.$$

Suppose this is false. Then v_k must tend to some constant K. Let us choose $\varepsilon > 0$, and suppose that for $m \geqq N$, $v_m \geqq K - \varepsilon$. Then for any $m \geqq N$, we can choose x_1, $\{y_r, z_r : r = 1, \cdots, k\}$, such that

$$\sum_{i=1}^{m} e_i \leqq x_1 + \sum_{r=1}^{k} \pi_r(y_r : z_r),$$

and

$$\|x_1\|_{l_1} + \sum_{r=1}^{k} (\|y_r\|_{c_0}\|z_r\| + \|z_r\|_{c_0}\|y_r\|) < K + \varepsilon.$$

Therefore there must be vectors y, z and a permutation π such that

$$\pi(y:z)\cdot(e_1 + \cdots + e_m) \geqq \frac{m}{K+\varepsilon}(\|y\|_{c_0}\|z\| + \|z\|_{c_0}\|y\|).$$

Suppose that y and z are two such vectors, say

$$\|y\|_{c_0} = \|z\|_{c_0} = 1.$$

If

$$\pi(y:z)\cdot(e_1 + \cdots + e_m) = \lambda, \qquad (\text{with } 2/(K+\varepsilon) \leqq \lambda \leqq 1)$$

then

$$\|z\| + \|y\| \leqq \lambda(K+\varepsilon).$$

Suppose

$$(*) \qquad \begin{aligned} \|z\| &= \delta\lambda(K+\varepsilon), \\ \|y\| &\leqq (1-\delta)\lambda(K+\varepsilon). \end{aligned}$$

There are at most $(N-1)$ coordinates of z which are greater than $\delta\lambda(K+\varepsilon)/(K-\varepsilon)$, and at most $(N-1)$ coordinates of y which are greater than $(1-\delta)\lambda(K+\varepsilon)/(K-\varepsilon)$. Therefore

$$\begin{aligned} \lambda_m &= \pi(y:z)\cdot(e_1 + \cdots + e_m) \\ &\leqq (N-1)^2 + (m-(N-1)^2)\lambda\frac{(K+\varepsilon)\max(\delta, 1-\delta)}{K-\varepsilon}. \end{aligned}$$

For small choice of ε, and large m, this is a contradiction unless δ or $1-\delta$ is very close to 1. But since $\|y\|, \|z\|$ are both greater than or equal to 1, in view of (*) we cannot have δ or $1-\delta$ less than $1/(K+\varepsilon)$. So in fact we cannot have $v_k \to K$, so v_k is unbounded.

We now define

$$\|x\|^{(N)} = \inf_{x_1+x_2=|x|} N\|x_1\|_{c_0} + \frac{1}{N}\|x_2\|.$$

LEMMA 1. *For each N, $\|\cdot\|^{(N)}$ also has the property that*

$$\|x : y\|^{(N)} \leqq \|x\|_{c_0}\|y\|^{(N)} + \|y\|_{c_0}\|x\|^{(N)}.$$

PROOF. Given x, y we may choose $\varepsilon > 0$, and put $\delta = \varepsilon/(\|x\|_{c_0} + \|y\|_{c_0})$. We can choose x_1, x_2, y_1, y_2 such that

$$x_1, x_2 \geqq 0; \qquad y_1, y_2 \geqq 0;$$

$$x_1 + x_2 = |x|, \qquad y_1 + y_2 = |y|;$$

and

$$\|x\|^{(N)} \geqq N\|x_1\|_{c_0} + \frac{1}{N}\|x_2\| - \delta,$$

$$\|y\|^{(N)} \geqq N\|y_1\|_{c_0} + \frac{1}{N}\|y_2\| - \delta.$$

Then $(x_1 + x_2) : (y_1 + y_2) = |x : y|$; therefore

$$x_1 : (y_1 + y_2) + (x_1 + x_2) : y_1 + x_2 : y_2 \geqq |x : y|,$$

and

$$\|x : y\|^{(N)} \leqq N\|x_1 : (y_1 + y_2)\|_{c_0} + N\|(x_1 + x_2) : y_1\|_{c_0} + \frac{1}{N}\|x_2 : y_2\|$$

$$\leqq N\|x_1\|_{c_0}\|y_1 + y_2\|_{c_0} + N\|y_1\|_{c_0}\|x_1 + x_2\|_{c_0}$$

$$+ \frac{1}{N}(\|x_2\|\|y_2\|_{c_0} + \|y_2\|\|x_2\|_{c_0}) \quad \text{(by the known properties of } \|\cdot\|)$$

$$\leqq (\|x\|^{(N)} + \delta)\|y\|_{c_0} + (\|y\|^{(N)} + \delta)\|x\|_{c_0}$$

$$\leqq \|x\|^{(N)}\|y\|_{c_0} + \|y\|^{(N)}\|x\|_{c_0} + \varepsilon.$$

Since this is true for any $\varepsilon > 0$, we have as required

$$\|x : y\|^{(N)} \leqq \|x\|^{(N)}\|y\|_{c_0} + \|y\|^{(N)}\|x\|_{c_0}.$$

The norms $\|\cdot\|^{(N)}$ have the following properties:

(1) $(1/N)\|x\|_{c_0} \leqq \|x\|^{(N)} \leqq N\|x\|_{c_0}$.

(2) $\|e_i\|^{(N)} = 1/N$.

(3) $\|e_1 + \cdots + e_k\|^{(N)} \to N$ as $k \to \infty$.

It follows that we may choose increasing sequences $(N_i)_{i=1}^{\infty}$, $(k_i)_{i=1}^{\infty}$, such that $N_1 = k_1 = 1$, and

$$\left| \left\| \frac{e_1 + \cdots + e_{k_i}}{N_i} \right\|^{(N_j)} - \delta_{ij} \right| < 2^{-(i+j)}.$$

Let us choose such a pair of sequences $(N_i)_{i=1}^{\infty}$ and $(k_i)_{i=1}^{\infty}$. Having done this, we define our spaces Y and Z.

§3. Definition and elementary properties of Y and Z

For each $x \in l_1$, define Tx to be that vector in l_1 whose ith coordinate is $\frac{1}{4}\|x\|^{(N_i)}$. Then

$$\begin{aligned}\|Tx\|_{l_1} &\leqq \|x\|_{l_1}\|Te_1\|_{l_1}\\ &\leqq \tfrac{1}{4}\|x\|_{l_1}\sum_{i=1}^{\infty}\|e_1\|^{(N_i)}\\ &\leqq \tfrac{1}{4}\|x\|_{l_1}\left(1+\sum_{i=1}^{\infty}2^{-i-1}\right)\\ &\leqq \tfrac{1}{2}\|x\|_{l_1}.\end{aligned}$$

Similarly, define Sx to be that vector in l_i whose ith coordinate is $\frac{1}{8}\cdot(\frac{3}{2})^i\|x\|^{(N_i)}$. As before,

$$\|Sx\|_{l_1} \leqq \|x\|_{l_1}\|Se_1\|_{l_1} \leqq \tfrac{1}{2}\|x\|_{l_1}.$$

Then we define Y to be the completion of l_1 under the norm

$$\begin{aligned}\|x\|_Y &= \|x\|_{c_0} + \|Sx\|_{c_0} + \|TSx\|_{c_0} + \|STSx\|_{c_0} + \cdots\\ &= \sum_{n=0}^{\infty}(\|(TS)^n x\|_{c_0} + \|S(TS)^n x\|_{c_0}),\end{aligned}$$

and we define Z to be the completion of l_1 under the norm

$$\|x\|_z = \|x\|_{c_0} + \|Tx\|_Y.$$

Then we see that, for any $x \in l_1$,

$$\|x\|_{c_0} \leqq \|x\|_Y \leqq 2\|x\|_{l_1},$$

and

$$\|x\|_{c_0} \leqq \|x\|_z \leqq 2\|x\|_{l_1}.$$

The unit vectors $(e_i)_{i=1}^{\infty}$ of l_1 now form a 1 symmetric basis of Y (or Z).

LEMMA 1. *On l_1, the Z norm is continuous with respect to the Y norm.*

For if $x = \sum_{i=1}^{\infty} \lambda_i e_i \in Y$, then for some M,

$$(\tfrac{3}{2})^i \| x \|^{(N_i)} \leqq M \qquad \text{for all } i \in \mathbf{N},$$

hence

$$\| x \|^{(N_i)} \leqq M(\tfrac{2}{3})^i;$$

therefore $Tx \in l_1$ and $x \in Z$.

But the Z-norm and the Y-norm are not equivalent; for

$$\left\| \frac{e_1 + \cdots + e_{k_i}}{N_i} \right\|_Y \geqq \frac{1}{16}\left(\frac{3}{2}\right)^i,$$

but

$$\left\| \frac{e_1 + \cdots + e_{k_i}}{N_i} \right\|_Z \leqq \frac{3}{2}.$$

Let $\sigma = (\sigma_j)_{j=1}^{\infty}$ be a sequence of consecutive disjoint finite subsets of the integers, such that $\bar{\bar{\sigma}}_j = k_j$ for all j. We have associated averaging projections $P_\sigma : Y \to Y$ and $Q_\sigma : Z \to Z$.

LEMMA 2. $P_\sigma Y \cong Z$; $Q_\sigma Z \cong Y$.

We prove that $Q_\sigma Z \cong Y$; the proof that $P_\sigma Y \cong Z$ is very similar. $Q_\sigma Z$ is the closed subspace of Z generated by vectors

$$v_i = \sum_{i \in \sigma_i} e_i.$$

Now $\bar{\bar{\sigma}}_i = k_i$, and we know that

$$\left| \left\| \frac{e_1 + \cdots + e_{k_i}}{N_i} \right\|^{(N_j)} - \delta_{ij} \right| < 2^{-(i+j)}.$$

So we can roughly normalise v_i by replacing it with

$$w_i = \frac{4 v_i}{N_i}.$$

We then have

$$\| T w_i - e_i \|_{l_1} = \sum_{j=1}^{\infty} \left| \frac{1}{4} \| w_i \|^{(N_j)} - \delta_{ij} \right|$$

$$\leqq \sum_{j=1}^{\infty} 2^{-(i+j)} = 2^{-i}.$$

Then, if $v = \Sigma_{i=1}^{r} \lambda_i w_i$,

$$\begin{aligned}\|v\|_Z &= \|v\|_{c_0} + \Big\| T\Big(\sum_{i=1}^{r} \lambda_i w_i\Big)\Big\|_Y \\ &\leqq 4\|\boldsymbol{\lambda}\|_{c_0} + \Big\|\sum_{i=1}^{r} \lambda_i e_i\Big\|_Y + \sum_{i=1}^{r} |\lambda_i| \, \|T(w_i) - e_i\|_Y \\ &\leqq 4\|\boldsymbol{\lambda}\|_{c_0} + \|\boldsymbol{\lambda}\|_Y + 2\|\boldsymbol{\lambda}\|_{c_0} \sum_{i=1}^{r} \|T(w_i) - e_i\|_{l_1} \\ &\leqq 4\|\boldsymbol{\lambda}\|_{c_0} + \|\boldsymbol{\lambda}\|_Y + \|\boldsymbol{\lambda}\|_{c_0} \sum_{i=1}^{r} 2^{-i+1} \\ &\leqq 6\|\boldsymbol{\lambda}\|_{c_0} + \|\boldsymbol{\lambda}\|_Y \leqq 7\|\boldsymbol{\lambda}\|_Y.\end{aligned}$$

However,

$$\begin{aligned}\|v\|_Z &\geqq \Big\| \bigvee_{i=1}^{r} |\lambda_i| \, T(w_i) \Big\|_Y \\ &\geqq \Big\| \sum_{i=1}^{r} |\lambda_i| \, e_i (1 - 2^{-2i}) \Big\|_Y \geqq \frac{3}{4} \|\boldsymbol{\lambda}\|_Y.\end{aligned}$$

So in fact $\{w_i\}_{i=1}^{\infty}$ is a basis for $Q_\sigma Z$, equivalent to the unit vector basis of Y.

Adding on the similar result $P_\sigma Y \cong Z$, and applying the argument of [1], section 1, we learn that $Y \cong Z$.

The rest of this paper is devoted to proving that, up to equivalence, there are no further symmetric bases of Y.

DEFINITION. Let $y \in Y$. We define

$$\omega_n(y) = \begin{cases} (TS)^{(n-1)/2} y, & \text{if } n \text{ is odd}, \quad n \geqq 3, \\ S(TS)^{(n-2)/2} y, & \text{if } n \text{ is even}, \quad n > 0, \\ |y|, & \text{if } n = 1. \end{cases}$$

Thus

$$\|y\|_Y = \sum_{n=1}^{\infty} \|\omega_n(y)\|_{c_0}.$$

LEMMA 3. *Let $(y_n)_{n=1}^{\infty}$ be a symmetric normalised block basis of Y. Suppose that $\Sigma_{r=k}^{\infty} \|\omega_r(y_n)\|_{c_0}$ does not tend to zero as $k \to \infty$ uniformly in n. Then $(y_n)_{n=1}^{\infty}$ is equivalent to the unit vector basis of l_1.*

PROOF. By hypothesis we may find $\varepsilon > 0$ and a subsequence $(y_{n_i})_{i=1}^{\infty}$ of $(y_i)_{i=1}^{\infty}$, and integers $m_1 < m_2 < \cdots$ such that, for all i,

$$\sum_{r=m_i}^{m_{i+1}-1} \|\omega_r(y_{n_i})\|_{c_0} \geqq \varepsilon.$$

Then

$$\begin{aligned}\|\lambda_1 y_{n_1} + \cdots + \lambda_k y_{n_k}\| &\geqq \sum_{r=1}^{m_k} \|\omega_r(\lambda_1 y_{n_1} + \cdots + \lambda_k y_{n_k})\|_{c_0} \\ &\geqq \sum_{i=1}^{k} \left(\sum_{r=m_i}^{m_{i+1}-1} |\lambda_i| \, \|\omega_r(y_{n_i})\|_{c_0} \right) \\ &\geqq \varepsilon \|\boldsymbol{\lambda}\|_{l_1}.\end{aligned}$$

So the subsequence is $1/\varepsilon$ equivalent to the unit vector basis of l_1; since the sequence is symmetric, this proves the lemma.

We now have to investigate all symmetric block bases of Y such that

$$\sum_{r=k}^{\infty} \|\omega_r(y_n)\|_{c_0} \xrightarrow{k \to \infty} 0,$$

uniformly in n.

§4. Ramsey theory

In this section I am going to state, without proof, two Ramsey type results.

DEFINITION. (a) Let Λ be the collection of all strictly increasing sequences $\boldsymbol{n} = (n_i)_{i=1}^{\infty}$ of natural numbers.

(b) If $\boldsymbol{m}, \boldsymbol{n} \in \Lambda$, say $\boldsymbol{m} \subset \boldsymbol{n}$ if $\{m_i\}_{i=1}^{\infty} \subset \{n_i\}_{i=1}^{\infty}$.

(c) If $\boldsymbol{m}, \boldsymbol{n} \in \Lambda$ define $\boldsymbol{m} \cdot \boldsymbol{n} = \mathbf{l}$ where $l_i = m_{n_i}$.

Λ may be regarded as a measurable subset of $2^{\mathbf{N}}$.

LEMMA 1. *If $C \subset \Lambda$ is measurable, then either*

(1) *there exists $\boldsymbol{n} \in \Lambda$ such that for all $\boldsymbol{m} \subset \boldsymbol{n}$, $\boldsymbol{m} \in C$; or*

(2) *there exists $\boldsymbol{n} \in \Lambda$ such that for all $\boldsymbol{m} \subset \boldsymbol{n}$, $\boldsymbol{m} \notin C$.*

DEFINITION. Let $Q : \Lambda \to \Lambda$ be the "shift" sending $(n_i)_{i=1}^{\infty}$ to $(m_i)_{i=1}^{\infty}$ where $m_i = n_{i+1}$.

We deduce from Lemma 1 the next lemma:

LEMMA 2. *Suppose $\tau : \Lambda \to K$ is measurable, where K is a compact metric space. Then we can find an $\boldsymbol{n} \in \Lambda$ and an $x \in K$ such that, for all $\boldsymbol{m} \subset \boldsymbol{n}$,*

$$\tau(Q'm) \xrightarrow[r\to\infty]{} x.$$

(This is essentially a repeated application of Lemma 1, together with a diagonal sequence argument.)

§5. Definition of convergence "permuted weakly"

DEFINITION. If $(x_i)_{i=1}^{\infty} \subset c_0$, we say

$$x_i \xrightarrow[i\to\infty]{p\omega} \alpha, \beta \qquad (\text{"permuted weakly"})$$

if $x_i \xrightarrow{\sigma(c_0, l_1)} \alpha$ and $(x_i - \alpha)^{\wedge} \xrightarrow{\sigma(c_0, l_1)} \beta$.

(Recall that, if $x \in c_0$, $\hat{x} = \pi(|x|)$, where π is chosen so that the coordinates of $\hat{x}$ decrease.)

We may think of this as saying that, for large i, x_i looks like α on the first few coordinates, but like a permutation of β on coordinates further on. This is made formal by

LEMMA 1. *If $x_i \xrightarrow{p\omega} \alpha, \beta$, we can find $\boldsymbol{n} \in \Lambda$ and $\pi \in S(\mathbf{N})$ such that, for each i,*

$$\| |x_{n_i} - \alpha| - \pi(e_i : \beta) \|_{c_0} < 2^{-i}.$$

PROOF. Without loss of generality $\alpha = 0$. Then since

$$x_i \xrightarrow{\omega} 0, \qquad \hat{x}_i \xrightarrow{\omega} \beta,$$

we may first choose a subsequence $\boldsymbol{m} \in \Lambda$ so that

$$\| \hat{x}_{m_i} - \beta \|_{c_0} < 2^{-i-2} \qquad (i = 1, 2, \cdots).$$

We may then choose a subsequence $\boldsymbol{n} \subset \boldsymbol{m}$, so that

$$\sigma_i = \{ j \in \mathbf{N} : | x_{n_i} \cdot e_j | > 2^{-i-2} \} \qquad (i = 1, 2, \cdots)$$

are a disjoint collection of finite subsets of $\mathbf{N}$; let us say

$$\max \sigma_i < \min \sigma_{i+1} - 1.$$

Then we can choose a permutation $\pi \in S(\mathbf{N})$ which, for each i, takes the largest $|\sigma_i|$ coordinates of $e_i : \beta$ onto the largest $|\sigma_i|$ coordinates of $|x_{m_i}|$; and then $\boldsymbol{n}$ and π will satisfy the condition of the lemma.

§6. Applying Ramsey theory to the problem in hand

First, some definitions.

Let

$$V = (c_0, \sigma(c_0, l_1))^{\mathbf{N}}.$$

Denote an element $v \in V$ by $(v_i)_{i=1}^{\infty}$ $(v_i \in c_0)$. Then we define $\phi : Y \to V$ by

$$y \to \{\omega_i(y)\}_{i=1}^{\infty}.$$

Then ϕ is continuous, since each ω_i is norm to norm continuous (in fact a contraction).

Now let Z_1 be the collection of all norm continuous maps $B(l_1) \to V$, with the topology of pointwise convergence. Z_1 is a locally convex space.

For $z \in Z_1$, define $\hat{z} \in Z_1$ by

$$(\hat{z}(\lambda))_i = ((z(\lambda))_i)^{\wedge}$$

for each $\lambda \in l_i$, $i \in \mathbf{N}$. "$\wedge$" is a measurable map $z_1 \to z_1$. (In fact, "$\wedge$" is not continuous since we have the weak topology on c_0.)

Let $(y_i)_{i=1}^{\infty}$ be a symmetric normalised block basis of Y. Then define $\tau : \Lambda \to Z_1$ by

$$\tau(\mathbf{n}) = T^{(\mathbf{n})} : B(l_1) \to V,$$

where

$$T^{(\mathbf{n})}\left(\sum_{i=1}^{\infty} \lambda_i e_i\right) = \phi\left(\sum_{i=1}^{\infty} \lambda_i y_{n_i}\right).$$

LEMMA 1. *τ is continuous and $\tau(\Lambda)$ is precompact.*

PROOF. (1) τ is continuous. For in Z_1 we have a subbase of neighbourhoods

$$\{z \in Z_1 : |\langle (z(\lambda))_i, u\rangle - \alpha| < 1\}$$

(where $\lambda, \mu \in l_1$, $\alpha \in \mathbf{R}$, and $(\ )_i$ denotes the ith component).

The inverse image of this under τ is

$$\left\{\mathbf{n} \in \Lambda : \left|\left\langle \omega_i\left(\sum_{i=1}^{\infty} \lambda_i y_{n_i}\right), u\right\rangle - \alpha\right| < 1\right\},$$

which is open in Λ because

$$\sum_{i=N}^{\infty} |\lambda_i| \xrightarrow[N \to \infty]{} 0.$$

(2) $\tau(\Lambda)$ is precompact. For for all $\boldsymbol{n} \in \Lambda$, $\lambda \in B(l_1)$, we have

$$T^{(\boldsymbol{n})}(\boldsymbol{\lambda}) \in (B(c_0))^{\mathbf{N}}$$

since $(y_i)_{i=1}^{\infty}$ is normalised.

But $(B(c_0))^{\mathbf{N}}$ is compact in V, so $\tau(\Lambda)$ is in a collection of uniformly continuous maps $B(l_1) \to K$, where $B(l_1)$ is separable metrisable, and K is compact. Thus $\tau(\Lambda)$ is precompact.

We may now apply to τ Lemma 2 of §4, and obtain a $z \in Z_1$, $\boldsymbol{n} \in \Lambda$ such that for all $\boldsymbol{m} \subset \boldsymbol{n}$,

$$\tau(Q^r\boldsymbol{m}) \xrightarrow{r \to \infty} z.$$

Then we define $\sigma : \Lambda \to Z_1$ by

$$\sigma(\boldsymbol{m}) = (\tau(\boldsymbol{m} \cdot \boldsymbol{n}) - z)^{\wedge}.$$

Then σ is measurable and $\sigma(\Lambda)$ is precompact. So, applying Lemma 2 again, we obtain $z' \in Z_1$ and $\boldsymbol{n}' \in \Lambda$ such that for all $\boldsymbol{m} \subset \boldsymbol{n}'$,

$$\sigma(Q^r\boldsymbol{m}) \xrightarrow[r \to \infty]{} z'.$$

Now z and z' are maps $B(l_1) \to V$. Since

$$V = (c_0, \sigma(c_0, l_1))^{\mathbf{N}},$$

we may consider the "components"

$$z_i : B(l_1) \to c_0 \qquad (i = 1, 2, \cdots)$$

and

$$z_i' : B(l_1) \to c_0 \qquad (i = 1, 2, \cdots),$$

such that

$$z(\lambda) = (z_i(\lambda))_{i=1}^{\infty} \quad \text{and} \quad z'(\lambda) = (z_i'(\lambda))_{i=1}^{\infty} \qquad (\lambda \in l_1).$$

We then have, for each $\lambda \in l_1$ and $\boldsymbol{m} \subset \boldsymbol{n}' \cdot \boldsymbol{n}$,

$$\omega_i\left(\sum_{j=1}^{\infty} \lambda_j y_{m_{j+r}}\right) \xrightarrow[r \to \infty]{p\omega} z_i(\lambda), z_i'(\lambda).$$

We wish to identify z_i and z_i' more closely; the following lemma enables us to identify z_i'.

LEMMA 2. *Suppose $\varphi : l_1 \to c_0$ is a contraction, and satisfies*

$$\phi(x) \geqq 0, \qquad \phi(\lambda x) = |\lambda| \phi(x) \qquad (\lambda \in \mathbf{R}, x \in l_1), \tag{1}$$

$$\phi(x_1 + x_2) \leqq \phi(x_1) + \phi(x_2) \qquad (x_1, x_2 \in l_1), \tag{2}$$

$$|x_1| \geqq |x_2| \Rightarrow \phi(x_1) \geqq \phi(x_2). \tag{3}$$

Suppose also that for all $\boldsymbol{m} \in \Lambda$,

$$\phi\Big(\sum_{j=1}^{\infty} \lambda_j e_{m_{j+r}}\Big) \xrightarrow[r \to \infty]{p\omega} z(\lambda), z'(\lambda).$$

Then

$$z'(\lambda) = (z'(e_1) : \lambda)^{\wedge}, \qquad \textit{for all } \lambda \in l_1.$$

PROOF. Put $x_i = \phi(e_i)$ and, applying Lemma 1 of §5, choose $\boldsymbol{m} \in \Lambda$ and $\pi \in S(\mathbf{N})$ such that, for each i,

$$\big\| |x_{m_i} - z(e_1)| - \pi(e_i : z'(e_1)) \big\|_{c_0} < 2^{-i}.$$

Since $x_i \geqq 0$ for all i, and $x_i \xrightarrow{\omega} z(e_1)$, we may assume that

$$\|(x_{m_i} - z(e_1))_-\|_{c_0} < 2^{-i-2},$$

so that

$$\|x_{m_i} - z(e_1) - \pi(e_i : \beta(e_1))\|_{c_0} < 2^{-i+1}.$$

In view of our hypotheses (2) and (3) we have, for all $\lambda \in l_1$,

$$\bigvee_{i=1}^{\infty} |\lambda_i| \phi(e_{m_i}) \leqq \phi\Big(\sum_{i=1}^{\infty} \lambda_i e_{m_i}\Big) \leqq \sum_{i=1}^{\infty} |\lambda_i| \phi(e_{m_i}).$$

So there are vectors ε_1 and ε_2 such that

$$\begin{aligned} \|\lambda\|_{c_0} z(e_1) + \pi(\lambda : z'(e_1)) - \varepsilon_1 &\leqq \phi\Big(\sum_{i=1}^{\infty} \lambda_i e_{m_i}\Big) \\ &\leqq \|\lambda\|_{l_1} z(e_1) + \pi(\lambda : z'(e_1)) + \varepsilon_2, \end{aligned}$$

with

$$\|\varepsilon_i\|_{c_0} \leqq \sum_{j=1}^{\infty} |\lambda_j| 2^{-j} \qquad (i = 1, 2).$$

But we know that

$$\phi\Big(\sum_{j=1}^{\infty} \lambda_j e_{m_{j+r}}\Big) \xrightarrow[r \to \infty]{p\omega} z(\boldsymbol{\lambda}), z'(\boldsymbol{\lambda}).$$

Therefore we must have

$$z'(\lambda) = (\lambda : z'(e_1))^\wedge,$$

as required.

In order to identify z_i we need a further lemma:

LEMMA 3. *Suppose* $x_i \in c_0$, $x_i \xrightarrow{p\omega} \alpha, \beta$. *Then for each* n,

$$\|x_i\|^{(n)} \longrightarrow \|[\alpha, \beta]\|^{(n)}$$

(*where* $[\alpha, \beta]$ *is as in part* (3)b *of* §1).

PROOF. $\|\cdot\|^{(n)}$ is equivalent to $\|\cdot\|_{c_0}$, and is 1-symmetric; but if $x_i \xrightarrow{p\omega} \alpha, \beta$, then, for large i, $|x_i|$ is c_0-norm close to a permutation of $[|\alpha|, |\beta|]$.

Now we return to our symmetric block basis $(y_i)_{i=1}^\infty$ of Y. By extracting a subsequence we may assume that for all $n \in \Lambda$,

$$\omega_i\left(\sum_{j=1}^\infty \lambda_j y_{n_j+r}\right) \xrightarrow[r\to\infty]{p\omega} z_i(\lambda), z_i'(\lambda).$$

In view of our first lemma, we know that there are elements $(\beta_i)_{i=1}^\infty$ such that

$$z_i'(\lambda) = (\beta_i : \lambda)^\wedge$$

for each i. It is obvious that $z_1(\lambda) = 0$ for all λ; and for each $i \geqq 1$, our second lemma shows that

$$z_{i+1}(\lambda) = \begin{bmatrix} T \\ S \end{bmatrix} [z_i(\lambda), z_i'(\lambda)],$$

where we take T if i is even and S if i is odd.

So we have identified z_i, z_i' as follows:

LEMMA 4. *There exists a collection* $(\beta_i)_{i=1}^\infty \subset c_0$, such that for all $\lambda \in l_1$,

(a) $z_1(\lambda) = 0$,

(b) $z_i'(\lambda) = (\beta_i : \lambda)^\wedge$ *for all* $i \geqq 1$,

(c) $z_{i+1}(\lambda) = \begin{bmatrix} T \\ S \end{bmatrix} [z_i(\lambda), z_i'(\lambda)]$ $(i \geqq 1)$,

where we take T *if* i *is even,* S *if* i *is odd.*

Now, for each i,

$$\omega_i\left(\sum_{j=1}^\infty \lambda_j y_{j+r}\right) \xrightarrow[r\to\infty]{p\omega} z_i(\lambda), z_i'(\lambda),$$

so for each fixed M,

$$\sum_{i=1}^{M}\left\|\omega_i\left(\sum_{j=1}^{\infty}\lambda_j y_{j+r}\right)\right\|_{c_0}\xrightarrow{r\to\infty}\sum_{i=1}^{M}\|z_i(\lambda)\|_{c_0}\vee\|z'_i(\lambda)\|_{c_0}.$$

Assume that $(y_i)_{i=1}^{\infty}$ is not equivalent to the unit vector basis of l_1. Then by Lemma 3 of §3,

$$\sum_{i=M+1}^{\infty}\|\omega_i(y_m)\|_{c_0}\xrightarrow{M\to\infty}0$$

uniformly in m. Therefore for each $\lambda\in l_1$,

$$\sum_{i=1}^{\infty}\left\|\omega_i\left(\sum_{j=1}^{\infty}\lambda_j y_{j+r}\right)\right\|_{c_0}\xrightarrow{r\to\infty}\sum_{i=1}^{\infty}\|z_i(\lambda)\|_{c_0}\vee\|z'_i(\lambda)\|_{c_0}.$$

But the left-hand side is $\|\sum_{j=1}^{\infty}\lambda_j y_{j+r}\|_Y$. Therefore, if our block basis is C-symmetric, it must induce a norm which is C-equivalent to

$$|||\lambda|||=\sum_{i=1}^{\infty}\|z_i(\lambda)\|_{c_0}\vee\|z'_i(\lambda)\|_{c_0}.$$

It is sufficient for our purposes to prove that this norm must be equivalent to either $\|\cdot\|_Y$ or $\|\cdot\|_Z$.

§7. Investigating $|||\cdot|||$

$|||\lambda|||$ depends on λ and the vectors $(\beta_i)_{i=1}^{\infty}$. We may thus regard $|||\cdot|||$ as a function

$$p:\ l_1\times c_0^{\mathbf{N}}\to\mathbf{R}^+\cup\{\infty\}$$

$$:(\lambda,(\beta_i)_{i=1}^{\infty})\to\sum_{i=1}^{\infty}\|z_i(\lambda)\|_{c_0}\vee\|z'_i(\lambda)\|_{c_0}$$

where z_i, z'_i are defined as in Lemma 4 of the previous section.

We shall show that $|||\cdot|||$ is equivalent to $\|\cdot\|_Y$ if there is a single $\beta_i\neq 0$ for odd i, and otherwise is equivalent to $\|\cdot\|_Z$.

Regarding $c_0^{\mathbf{N}}$ as a vector lattice, we observe that

$$|\boldsymbol{\beta}_1|<|\boldsymbol{\beta}_2|\quad\text{implies}\quad p(\lambda,\boldsymbol{\beta}_1)\leqq p(\lambda,\boldsymbol{\beta}_2),$$

$$p(\lambda,\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)\leqq p(\lambda,\boldsymbol{\beta}_1)+p(\lambda,\boldsymbol{\beta}_2),$$

for all $\lambda\in l_1$, $\boldsymbol{\beta}_1,\boldsymbol{\beta}_2\in c_0^{\mathbf{N}}$.

Let $\pi: c_0^{\mathbf{N}}\to c_0^{\mathbf{N}}$ be the projection onto the odd coordinates;

$$\pi((\beta_i)_{i=1}^{\infty}) = (\beta'_i)_{i=1}^{\infty},$$

where

$$\beta'_i = \begin{cases} \beta_i & \text{if } i \text{ is odd,} \\ 0 & \text{if } i \text{ is even.} \end{cases}$$

Then, for all $\boldsymbol{\beta} \in c_0^{\mathbf{N}}$, $\lambda \in l_1$,

$$p(\lambda, \pi\boldsymbol{\beta}) \vee p(\lambda, (1-\pi)\boldsymbol{\beta}) \leqq p(\lambda, \boldsymbol{\beta}) \leqq p(\lambda, \pi\boldsymbol{\beta}) + p(\lambda, (1-\pi)\boldsymbol{\beta}).$$

It is thus sufficient to prove

Lemma 1.
(1) *If* $\pi\boldsymbol{\beta} \neq 0$, $p(\boldsymbol{\lambda}, \pi\boldsymbol{\beta})$ *is equivalent to* $\|\boldsymbol{\lambda}\|_Y$.
(2) *If* $(1-\pi)\boldsymbol{\beta} \neq 0$, $p(\boldsymbol{\lambda}, (1-\pi)\boldsymbol{\beta})$ *is equivalent to* $\|\boldsymbol{\lambda}\|_Z$.

We prove part (1); part (2) is entirely similar. Firstly, we have

$$\text{(a)} \qquad p(\lambda, \pi\boldsymbol{\beta}) \geqq \left(\sup_i \|\beta_{2i-1}\|_{c_0}\right) \|\lambda\|_Y.$$

In view of (*) above, the worst case is when

$$\beta_1 = C \cdot e_1, \quad \beta_{2n+1} = 0 \qquad \text{for all } n \geqq 1.$$

But in this case we have equality.

Second, and harder, is

$$\text{(b)} \qquad p(\lambda, \pi\boldsymbol{\beta}) \leqq p(e_1, \pi\boldsymbol{\beta}) \cdot \|\lambda\|_Y.$$

Here at last we use our inequality

$$\|x : y\|^{(N)} \leqq \|x\|_{c_0} \|y\|^{(N)} + \|y\|_{c_0} \|x\|^{(N)}$$

(Lemma 1 of §2). This implies that

$$S(x : y) \leqq \|x\|_{c_0} Sy + \|y\|_{c_0} Sx,$$

and

$$T(x : y) \leqq \|x\|_{c_0} Ty + \|y\|_{c_0} Tx.$$

Let z_i and z'_i ($i = 1$ to ∞) be the functions such that

$$p(\lambda, \pi\boldsymbol{\beta}) = \sum_{i=1}^{\infty} \|z_i(\lambda)\|_{c_0} \vee \|z'_i(\lambda)\|_{c_0},$$

and put

$$\varepsilon_i = \| z_i(e_1) \|_{c_0} \vee \| z'_i(e_1) \|_{c_0} \qquad (i = 1, 2, \cdots).$$

I claim that

$$z_i(\lambda) \leqq \| \lambda \|_{c_0} z_i(e_1) + \sum_{0 \leqq 2j \leqq i-2} \omega_{i-2j}(\lambda) \cdot \varepsilon_{2j+1}.$$

This is proved by induction on i, advancing two steps at a time. The result is true for $i = 1$. Suppose the result holds for $i = 2k - 1$. Then

$$\begin{aligned} z_{2k}(\lambda) &= S([z_{2k-1}(\lambda), \beta_{2k-1} : \lambda]) \\ &\leqq S\Big(\Big[\Big(\| \lambda \|_{c_0} z_{2k-1}(e_1) + \sum_{0 \leqq j \leqq k-2} \omega_{2k-2j-1}(\lambda) \cdot \varepsilon_{2j+1}\Big), \beta_{2k-1} : \lambda\Big]\Big) \\ &\leqq S([z_{2k-1}(e_1), \beta_{2k-1}] : \lambda) + \sum_{0 \leqq j \leqq k-2} S\omega_{2k-2j-1}(\lambda) \cdot \varepsilon_{2j+1} \\ &\leqq \| \lambda \|_{c_0} S[z_{2k-1}(e_1), \beta_{2k-1}] + \varepsilon_{2k-1} S\lambda + \sum_{0 \leqq j \leqq k-2} S\omega_{2k-2j-1}(\lambda) \cdot \varepsilon_{2j+1} \\ &= \| \lambda \|_{c_0} z_{2k}(e_1) + \sum_{0 \leqq 2j \leqq 2k-2} \omega_{2k-2j}(\lambda) \cdot \varepsilon_{2j+1}. \end{aligned}$$

This is the result for $i = 2k$. Then

$$\begin{aligned} z_{2k+1}(\lambda) &= T z_{2k}(\lambda) \qquad \text{(there is no contribution from } \beta_{2k}) \\ &\leqq \| \lambda \|_{c_0} T z_{2k}(e_1) + \sum_{0 \leqq 2j \leqq 2k-2} T\omega_{2k-2j}(\lambda) \cdot \varepsilon_{2j+1} \\ &= \| \lambda \|_{c_0} z_{2k+1}(e_1) + \sum_{0 \leqq 2j \leqq 2k-1} \omega_{2k+1-2j}(\lambda) \cdot \varepsilon_{2j+1}; \end{aligned}$$

so by induction, the claim is true for all i.

But now we are home. For

$$\begin{aligned} p(\lambda, \pi\boldsymbol{\beta}) &= \sum_{i=1}^{\infty} \| z_i(\lambda) \|_{c_0} \vee \| z'_i(\lambda) \|_{c_0} \\ &\leqq \sum_{i=1}^{\infty} \| \lambda \|_{c_0} (\| z_i(e_1) \|_{c_0} \vee \| z'_i(e_1) \|_{c_0}) + \sum_{i=1}^{\infty} \sum_{0 \leqq 2j \leqq i-2} \| \omega_{i-2j}(\lambda) \|_{c_0} \cdot \varepsilon_{2j+1} \\ &\leqq \sum_{i=1}^{\infty} \| \omega_i(\lambda) \|_{c_0} \cdot \sum_{j=1}^{\infty} \varepsilon_j \\ &= \| \lambda \|_Y p(e_1, \pi\boldsymbol{\beta}). \end{aligned}$$

Given the similar result

$$(\sup \| \beta_{2_i} \|_{c_0}) \| \lambda \|_Z \leqq p(\lambda, (1-\pi)\boldsymbol{\beta}) \leqq p(e_1, (1-\pi)\boldsymbol{\beta}) \| \lambda \|_Z,$$

we know that $p(\lambda, \boldsymbol{\beta})$ is either equivalent to $\| \cdot \|_Y$ or $\| \cdot \|_Z$, depending on whether $\pi\boldsymbol{\beta} = 0$. But at the end of §6 we deduced that any symmetric block basis of Y which is not equivalent to the unit vector basis of l_1, must induce a norm which is equivalent to

$$\||\lambda\|| = p(\lambda, \boldsymbol{\beta})$$

for some $\boldsymbol{\beta}$. Thus any symmetric basic sequence in Y is equivalent either to the unit vector basis of l_1, or to that of Y, or to that of Z. So Y has just the two symmetric bases.

§8. Banach spaces with any finite number of symmetric bases, or countably many

This section does not contain formal proofs. First, let us consider our maps S and $T: l_1 \to l_1$, and produce some others with similar properties.

Recall that Tx is that vector in l_1 whose ith coordinate is $\frac{1}{4}\| x \|^{(N_i)}$, and Sx is that vector in l_1 whose ith coordinate is $\frac{1}{8} \cdot (\frac{3}{2})^i \cdot \| x \|^{(N_i)}$.

We may similarly define, for $1 \leqq r < 2$,

$$S_r(x) = K_r \cdot \sum_{i=1}^{\infty} e_i \cdot \| x \|^{(N_i)} \cdot r^i.$$

If the normalisation constants K_r are chosen correctly, we will have

$$\| S_r e_1 \|_{l_1} = \frac{1}{2} = \sup_{x \in l_1} \frac{\| S_r x \|_{l_1}}{\| x \|_{l_1}}.$$

We will also have, for each $r \in [1, 2)$,

$$S_r(x:y) \leqq \| x \|_{c_0} S_r(y) + \| y \|_{c_0} S_r(x).$$

In order to produce a space with N symmetric bases, we take N different real numbers $r_1, \cdots, r_n$ from $[1, 2)$, and define our space Y to be the completion of l_1 under the norm

$$\begin{aligned} \| x \|_Y &= \| x \|_{c_0} + \| S_{r_1} x \|_{c_0} + \| S_{r_2} S_{r_1} x \|_{c_0} + \cdots \\ &\quad + \| S_{r_n} \cdots S_{r_1} x \|_{c_0} + \cdots + \| S_{r_k} S_{r_{k-1}} \cdots S_{r_1} (S_{r_n} \cdots S_{r_1})^j x \|_{c_0} + \cdots \\ &= \sum_{j=0}^{\infty} \sum_{i=0}^{n-1} \| (S_{r_i} \cdot S_{r_{i-1}} \cdot \cdots \cdot S_{r_1}) \cdot (S_{r_n} \cdot \cdots \cdot S_{r_1})^j x \|_{c_0}. \end{aligned}$$

Then, by methods precisely analogous to the case $N = 2$, we discover that Y has, up to equivalence, precisely N symmetric bases.

In order to do the countably infinite case, we need to be slightly more subtle. In this case, we choose a strictly descending sequence $r_0, r_1, r_2, \cdots$, from $[1,2)$, and put $S = S_{r_0}$ and $T_n = S_{r_n}$ $(n \geqq 1)$.

Also, we split up the natural numbers into a countable collection of disjoint infinite subsets $(H_n)_{n=1}^{\infty}$, and define $\pi_n : l_1 \to l_1$ to be the natural projection onto the coordinates in the nth subset H_n.

We shall define our new space Y as the completion of l_1 with a new norm $\|\cdot\|_Y$, which is, as before, an infinite sum of the supremum norms of certain vectors.

In the earlier example of a space with two symmetric bases, the collection of vectors used was

$$x, Sx, TSx, \cdots, (TS)^n x, S(TS)^n x, \cdots.$$

In this case, however, the Y norm is to be the sum of the supremum norms of all vectors in the following collection:

$$x, (\pi_i Sx)_{i=1}^{\infty}, (T_i \pi_i Sx)_{i=1}^{\infty}, (\pi_j S T_i \pi_i Sx)_{i,j=1}^{\infty}, (T_j \pi_j S T_i \pi_i Sx)_{i,j=1}^{\infty}, \cdots,$$

that is, the collection of all vectors of form

$$\prod_{k=1}^{m} (T_{i_k} \pi_{i_k} S)x \qquad (i_1, \cdots, i_m \in \mathbf{N})$$

or of form

$$\pi_{i_{k+1}} S \cdot \prod_{k=1}^{m} (T_{i_k} \pi_{i_k} S)x$$

as m runs from 0 to infinity.

What happens is, we take the supremum of the vector x, and then split up the sequence Sx into a countable number of disjoint infinite subsequences, and add on the suprema of each of these "bits". We then apply T_i to the ith "bit", and obtain an infinite collection of new vectors; and we repeat the whole process for each one of the new vectors in turn (adding on the suprema, applying S and splitting up, adding on all the new suprema, applying an appropriate T_i).

We continue the process inductively; and this gives us a new norm $\|\cdot\|_Y$ which satisfies

$$\|x\|_Y = \|x\|_{c_0} + \sum_1^{\infty} \|\pi_i Sx\|_{c_0} + \sum_1^{\infty} \|T_i \pi_i Sx\|_Y.$$

(Incidentally, it is still true that this norm is bounded by twice the l_1 norm.) So if we define new norms $\|\cdot\|_{z_i}$, also symmetric, by

$$\|x\|_{z_i} = \|x\|_{c_0} + \|T_i x\|_Y,$$

then we have

$$\|x\|_Y = \|x\|_{c_0} + \sum_1^\infty \|\pi_i S x\|_{z_i}.$$

Now the Z_i norms, $i = 1$ to ∞, correspond to the countable collection of alternative symmetric bases which Y contains. In fact, using our earlier notation (at the beginning of §1), if we choose $n \in \mathbf{N}$ and take an averaging projection on Y whose lengths $\bar{\bar{\sigma}}_j$ correspond to those k_i (see the end of §2) such that $i \in H_n$; then the image of the averaging projection is isomorphic to Z_n. But, in Z_n, if we take an averaging projection with $\bar{\bar{\sigma}}_j = k_j$, we have an image isomorphic to Y again. Thus $Y \cong Z_n$ for all n. So we have found our countable collection of symmetric bases; and, as before, we wish to show that there are no more. (This is where the proof becomes very informal.)

Now, as in the case of two symmetric bases, the space we have obtained contains copies of l_1; and if a symmetric basic sequence $(y_i)_{i=1}^\infty$ is not equivalent to the unit vector basis of l_1, then the amount of weight in the "tail" of the sequence of supremum norms defining $\|y_i\|$ must tend to zero uniformly (compare Lemma 3, §3). So, as before, we can assume that there is a limit p.w. of the vectors which defines the Y norm of $\Sigma_1^\infty \lambda_i y_i$ ($\boldsymbol{\lambda} \in l_1$).

If these limits are $(\alpha_k(\lambda), \beta_k(\lambda))$ (as k runs through the integers and $(\alpha_k(\lambda), \beta_k(\lambda))$ run through the p.w. limits of all the vectors defining the norm), then

$$\beta_k(\lambda) = (\lambda : \beta_k(e_1))^\wedge \quad \text{and} \quad \alpha_k(\lambda) = \begin{bmatrix} \pi_i S \\ T_i \end{bmatrix} [\alpha_l(\lambda), \beta_l(\lambda)],$$

where $\pi_i S$ or T_i and i are chosen appropriately.

Now the maps $\pi_i S$; $i \in \mathbf{N}$ and T_i still satisfy

$$T_i(x : y) \leqq \|x\|_{c_0} T_i y + \|y\|_{c_0} T_i x$$

and

$$\pi_i S(x : y) \leqq \|x\|_{c_0} \pi_i S y + \|y\|_{c_0} \pi_i S x.$$

Using arguments similar to those of §7, we can find an upper bound on $\|\Sigma_1^\infty \lambda_i y_i\|$ which, within the symmetric constant of the sequence, turns out to be

$\sum_{k=1}^{\infty} \| \alpha_k(e_1), \beta_k(e_1) \|_{c_0} \cdot \| \lambda \|_{(k)}$ where for $\| \lambda \|_{(k)}$ we take $\| \lambda \|_{c_0}$ if $\beta_k(e_1) = 0$, but we take the appropriate Y or Z_i norm for the value k if $\beta_k(e_1) \neq 0$. But

$$\sum_{1}^{\infty} \| \alpha_k(e_1), \beta_k(e_1) \|_{c_0} < \infty$$

or else $\| y_i \|$ would be unbounded. And now we use the fact that our norms $\| \cdot \|_{Z_i}$ are essentially descending (in other words, the norms $\| \cdot \|_{Z_{i+1}}, \| \cdot \|_{Z_{i+2}}, \| \cdot \|_{Z_{i+3}}, \cdots$ are uniformly continuous with respect to $\| \cdot \|_{Z_i}$).

For then, if $\beta_k \neq 0$ at a point corresponding to $\| \cdot \|_Y$, then the sequence is equivalent to the unit vector basis of Y; and otherwise it is equivalent to the unit vector basis of Z_i, where i is the smallest value (hence, the largest norm) for which there is an appropriate $\beta_k(e_1) \neq 0$.

Thus Y has a countably infinite number of symmetric bases.

§9. Notes

There is a proof of Lemma 1 of §4 (Ramsey theory) in [2]. The existence of Orlicz sequence spaces, with uncountably many mutually non-equivalent symmetric bases, is proved in chapter 4 of [4] (see p. 153), and also in [3].

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